Updated February 23, 2026

Homework problems for AMAT 300 (Intro to Proofs), Spring 2026. Over the course of the semester I'll add problems to this list, with each problem's due date specified. Each problem is worth 2 points.

Solutions will be gradually added (and may be hastily written without proofreading).

Problem 1 (due Weds 2/4): Let $A = \{20a - 26b \mid a, b \in \mathbb{Z}\}$ and $B = \{2c \mid c \in \mathbb{Z}\}$. Prove that $A = B$. (Do it directly, don't use any number theory tricks you might happen to know.)

Solution: ($\subseteq$): Let $x \in A$, so $x = 20a - 26b$ for some $a, b \in \mathbb{Z}$. Set $c = 10a - 13b$, so $c \in \mathbb{Z}$. Now $x = 2c$, so $x \in B$.
($\supseteq$): Let $x \in B$, so $x = 2c$ for some $c \in \mathbb{Z}$. Set $a = 4c$ and $b = 3c$, so $a, b \in \mathbb{Z}$. Then $x = 2c = 80c - 78c = 20(4c) - 26(3c) = 20a - 26b$, so $x \in A$. $\square$

Problem 2 (due Weds 2/4): Prove that the cardinality of $\{1, \{1\}, \{1, \{1\}\}, \{1, 1\}, \{\{1\}, 1\}\}$ is three. (You need to argue that it has exactly three elements, no more, no less.)

Solution: Since $\{\{1\}, 1\} = \{1, \{1\}\}$ and $\{1, 1\} = \{1\}$, we know that our set equals $\{1, \{1\}, \{1, \{1\}\}\}$, hence has at most 3 elements. Also, $1 \neq \{1\}$ since the latter is a set (with one element) and the former is not, which implies $\{1, \{1\}\}$ has two elements, and we conclude that these three elements are distinct. Thus the original set has exactly 3 elements. $\square$

Problem 3 (due Weds 2/4): Let A and B be non-empty finite sets such that $|\mathcal{P}(A \times B)| = |\mathcal{P}(A) \times \mathcal{P}(B)|$. Prove that A and B each have exactly two elements. (Careful not to *assume* they each have exactly two elements, you have to *prove* that they do.)

Solution: Set $a = |\mathcal{P}(A)|$ and $b = |\mathcal{P}(B)|$. Since $|\mathcal{P}(X)| = 2^{|X|}$ and $|X \times Y| = |X||Y|$ for any finite sets X and Y, we compute that $2^{ab} = 2^a 2^b$, hence $2^{ab} = 2^{a+b}$. Taking $\log_2$ we get $ab = a + b$. Since $a, b \in \mathbb{N}$, dividing by a yields $b = 1 + \frac{b}{a}$ and dividing by b yields $a = \frac{a}{b} + 1$, so $\frac{a}{b}$ and $\frac{b}{a}$ are both integers, which implies $a = b$. Now we solve $a^2 = 2a$ to get $a = b = 2$. $\square$

Problem 4 (due Weds 2/11): Let A and B be sets such that $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$ and suppose there exists $b_0 \in B \setminus A$. Prove that $A \subseteq B$.

Solution: Let $a \in A$. Then $\{a, b_0\}$ is a subset of $A \cup B$, hence an element of $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$. Since $b_0 \notin A$ we know $\{a, b_0\}$ is not in $\mathcal{P}(A)$, so it must be in $\mathcal{P}(B)$, i.e., $\{a, b_0\} \subseteq B$. We conclude that $a \in B$ as desired. $\square$

Problem 5 (due Weds 2/11): Let $\Lambda = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$ be the closed disk of radius 1 centered at the origin. For each $\alpha \in \Lambda$, say $\alpha = (x_0, y_0)$, let $X_\alpha = \{(x,y) \in \mathbb{R}^2 \mid (x-x_0)^2+(y-y_0)^2 \leq 1\}$ be the closed disk of radius 1 centered at α. Prove (rigorously) that $\bigcup_{\alpha\in\Lambda} X_\alpha = D$, where $D := \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4\}$ is the closed disk of radius 2 centered at the origin. [Note: I had a typo initially, writing $= 1$ instead of ≤ 1 in the definition of X_α, but the description "closed disk of radius 1 centered at α" made it clear what I meant and it seems no one got too confused.]

Solution: ($\subseteq$): Let $(x,y) \in \bigcup_{\alpha\in\Lambda} X_\alpha$, so $(x,y) \in X_\alpha$ for some $\alpha \in \Lambda$, say $\alpha = (x_0,y_0)$. Since $\alpha \in \Lambda$ we have $d((0,0),\alpha) \leq 1$, and since $(x,y) \in X_\alpha$ we have $d(\alpha,(x,y)) \leq 1$ (here d is the usual distance in $\mathbb{R}^2$). Now $d((0,0),(x,y)) \leq d((0,0),\alpha) + d(\alpha,(x,y))$ (by the triangle inequality, which is really just that the distance between two points is the shortest length of a path between them), and this is at most 2, so $(x,y) \in D$.
($\supseteq$): Let $(x,y) \in D$, so $d((0,0),(x,y)) \leq 2$. If $(x,y) \leq 1$ then $(x,y) \in \Lambda$ and $(x,y) \in X_{(x,y)}$ so we are done. Now assume $1 < (x,y) \leq 2$. Let $\alpha = (x,y)/d((x,y),(0,0))$, so $d(\alpha,(0,0)) = 1$, which shows $\alpha \in \Lambda$. Moreover, $d((x,y),\alpha) = d((x,y),(0,0)) - d(\alpha,(0,0)) \leq 2-1 = 1$, so $(x,y) \in X_\alpha$. This shows $(x,y) \in \bigcup_{\alpha\in\Lambda} X_\alpha$, and we are done. □

Problem 6 (due Weds 2/11): With the same setup as the previous problem, prove (rigorously) that $\bigcap_{\alpha\in\Lambda} X_\alpha = \{(0,0)\}$.

Solution: ($\subseteq$): Let $(x,y) \in \bigcap_{\alpha\in\Lambda} X_\alpha$, so $(x,y) \in X_\alpha$ for all $\alpha \in \Lambda$. In particular (x,y) is in $X_{(-1,0)}$ and $X_{(1,0)}$, so $(x+1)^2+y^2 \leq 1$ and $(x-1)^2+y^2 \leq 1$. Expanding these and adding them we get $2x^2+2+2y^2 \leq 2$, hence $x^2+y^2 \leq 0$, which means $x = y = 0$.
($\supseteq$): We must show that $(0,0) \in X_\alpha$ for all $\alpha \in \Lambda$. The criterion for α to be in Λ is $d((0,0),\alpha) \leq 1$, so this is immediate. □

Problem 7 (due Weds 2/18): Construct truth tables for $\sim(P \Leftrightarrow Q)$ and $(P \vee Q) \wedge (\sim(P \wedge Q))$ (showing the intermediate steps), and compare them. (Which logical operation do these represent?)

Solution: Hard to type out truth tables, you get F-T-T-F in both cases, so they're the same, and they represent "exclusive or". □

Problem 8 (due Weds 2/18): Prove that $n \in \mathbb{Z}$ is a multiple of 24 only if it is a multiple of 4 and 6. (Careful, this is not an "iff".)

Solution: Suppose $n \in \mathbb{Z}$ is a multiple of 24, say $n = 24m$ for some $m \in \mathbb{Z}$. Then $n = 4(6m)$ and $n = 6(4m)$, and since $6m, 4m \in \mathbb{Z}$ we conclude that n is a multiple of 4 and 6. □

Problem 9 (due Weds 2/18): Let X be a set. Prove that $|\mathcal{P}(X)| \leq 31$ if and only if $|X| \leq 4$.

Solution: ($\Rightarrow$): Suppose $|\mathcal{P}(X)| \leq 31$, so $2^{|X|} \leq 31$. Since $2^{|X|}$ is 2 to the power of a whole number, in fact $2^{|X|} \leq 16$. Thus $|X| \leq 4$.
($\Leftarrow$): Suppose $|X| \leq 4$. Then $2^{|X|} \leq 16$, so $|\mathcal{P}(X)| \leq 16$, and hence $|\mathcal{P}(X)| \leq 31$. $\square$

(Reminder: There's an exam in class on Weds 2/25.)
Problem 10 (due Weds 2/25): Let P and Q be statements. Prove that $\sim (P \Rightarrow (\sim Q))$ is logically equivalent to $(P \Leftrightarrow Q) \wedge (P \vee Q)$.

Problem 11 (due Weds 2/25): Let P be the statement, "every set with cardinality 2026 is a subset of a set with cardinality 2027." Convert P into a purely logical statement (using $\forall$, $\exists$, and so forth), and compute its negation. Then prove that P is true.

Problem 12 (due Weds 2/25): Let P be the statement, "every natural number divisible by its own square is divisible by at least two natural numbers." Convert P into a purely logical statement, and prove that P is false.

Nothing due Weds 3/4

Problem 13 (due Weds 3/11):

Problem 14 (due Weds 3/11):

Problem 15 (due Weds 3/11):
