Updated November 21, 2024

Homework problems for AMAT 540A (Topology I), Fall 2024. Over the course of the semester I'll add problems to this list, with each problem's due date specified. Each problem is worth 2 points.

Solutions will be gradually added (and may be hastily written without proofreading).

Problem 1 (due Weds 9/4): Let A be a set and let R be a relation on A that is reflexive, symmetric, transitive, and antisymmetric. Prove that for all $a, a' \in A$ we have aRa' if and only if $a = a'$.

Solution: ($\Rightarrow$): Suppose aRa'. By symmetry, $a'Ra$. By antisymmetry, $a = a'$. ($\Leftarrow$): Suppose $a = a'$. By reflexivity, aRa. Since $a = a'$, we conclude aRa'. $\square$

Problem 2 (due Weds 9/4): Let $f\colon A \to B$ be a function. Suppose that for all $A_0 \subseteq A$ we have $f^{-1}(f(A_0)) = A_0$. Prove that f is injective.

Solution: Let $a, a' \in A$ such that $f(a) = f(a')$. Then

$$\{a\} = f^{-1}(f(\{a\})) = f^{-1}(\{f(a)\}) = f^{-1}(\{f(a')\}) = f^{-1}(f(\{a'\})) = \{a'\},$$

so $a = a'$. $\square$

Problem 3 (due Weds 9/4): Prove that if a relation is symmetric and antisymmetric, then it is transitive. (So the thing I couldn't think of in class actually doesn't exist, which is nice.)

Solution: Suppose R is a relation on a set A, and R is symmetric and antisymmetric. Let $a, b, c \in A$ such that aRb and bRc. By symmetry, cRb. By antisymmetry, $b = c$. Since aRb and $b = c$ we conclude aRc. $\square$

Problem 4 (due Weds 9/11): Let X be a set and view the power set $\mathcal{P}(X)$ as a poset via the partial order $\subseteq$. Prove that every subset $\mathcal{A}$ of $\mathcal{P}(X)$ has a greatest lower bound. [Added later: If you want to assume $\mathcal{A}$ is non-empty that's fine, the $\mathcal{A} = \emptyset$ case is a little weird. (Technically *every* element is a lower bound of the empty set, but that's weird.)]

Solution: Let $\mathcal{A} \subseteq \mathcal{P}(X)$. Let $B = \bigcap_{A \in \mathcal{A}} A$. Then $B \subseteq A$ for all $A \in \mathcal{A}$, so B is a lower bound of $\mathcal{A}$. Also, given any lower bound C of $\mathcal{A}$, we have $C \subseteq A$ for all $A \in \mathcal{A}$, so $C \subseteq B$. We conclude that B is the greatest lower bound of $\mathcal{A}$. [Remark: If $\mathcal{A} = \emptyset$ then, vacuously, every subset of X is a lower bound of $\emptyset$, and so X itself is the greatest lower bound of $\emptyset$. Weird.] $\square$

Problem 5 (due Weds 9/11): Let P and Q be posets with partial orders $\le_P$ and $\le_Q$. Let $\le_{prod}$ be the product order on $P \times Q$. Prove that if every subset of P has a least upper bound in $\le_P$ and every subset of Q has a least upper bound in $\le_Q$, then every subset of $P \times Q$ has a least upper bound in $\le_{prod}$.

Solution: Let $X \subseteq P \times Q$. Let $A = \{p \in P \mid (p,q) \in X \text{ for some } q \in Q\}$ and $B = \{q \in Q \mid (p,q) \in X \text{ for some } p \in P\}$. Let a be the least upper bound of A and let b be the least upper bound of B. We claim that (a,b) is the least upper bound of X. For any $(p,q) \in X$ we have $p \le_P a$ and $q \le_Q b$, so $(p,q) \le_{prod} (a,b)$, so it is an upper bound. Now let (c,d) be any upper bound of X. For any $(p,q) \in X$ we have $p \le_P c$ and $q \le_Q d$. Since a and b are the respective least upper bounds, $a \le_P c$ and $b \le_Q d$. Thus $(a,b) \le_{prod} (c,d)$, and we conclude that (a,b) is the least upper bound of X. $\square$

Problem 6 (due Weds 9/11): Consider the Cantor set $C = \prod_{\mathbb{N}}\{0,1\}$, so elements of C are infinite binary sequences $(a_1, a_2, \dots)$. Let $\psi\colon C \to \mathbb{R}$ be the function sending $(a_1, a_2, \dots)$ to $\sum_{i=1}^{\infty} \frac{a_i}{10^i}$ (don't worry, this infinite series converges thanks to some calculus thing, so this really is an element of $\mathbb{R}$). Prove that ψ is injective.

Solution: Let $\vec{a} = (a_1, a_2, \dots), \vec{b} = (b_1, b_2, \dots) \in C$ such that $\psi(\vec{a}) = \psi(\vec{b})$, so $\sum_{i=1}^{\infty} \frac{a_i}{10^i}$ equals $\sum_{i=1}^{\infty} \frac{b_i}{10^i}$. By calculus stuff, $\sum_{i=1}^{\infty} \frac{a_i - b_i}{10^i} = 0$. Multiplying both sides by 10 we get $a_1 - b_1 + \sum_{i=1}^{\infty} \frac{a_{i+1} - b_{i+1}}{10^i} = 0$. Since all the a_i and b_i are 0 or 1, we know that $\sum_{i=1}^{\infty} \frac{a_{i+1} - b_{i+1}}{10^i}$ is bounded above and below by $\sum_{i=1}^{\infty} \frac{1}{10^i} = 1/9$ and $\sum_{i=1}^{\infty} \frac{-1}{10^i} = -1/9$. Hence $|a_1 - b_1| \le 1/9$, so $a_1 - b_1 = 0$. Now multiplying our new equation $\sum_{i=1}^{\infty} \frac{a_{i+1} - b_{i+1}}{10^i} = 0$ on both sides by 10, we get $a_2 - b_2 + \sum_{i=1}^{\infty} \frac{a_{i+2} - b_{i+2}}{10^i} = 0$, and the same argument shows $a_2 - b_2 = 0$. Repeating this, we get $a_i - b_i = 0$ for all i, and conclude that $\vec{a} = \vec{b}$. $\square$

Problem 7 (due Weds 9/18): Let $X = \{a, b\}$ and let $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfy $\emptyset, X \in \mathcal{T}$. Prove that $\mathcal{T}$ is a topology.

Solution: Since X is finite and $\mathcal{T}$ contains $\emptyset$ and X, it suffices to show that for any $U, V \in \mathcal{T}$ we have $U \cup V, U \cap V \in \mathcal{T}$. If either U or V is $\emptyset$ or X, or if $U = V$, this is trivially true. Now assume none of these are the case, so U and V each have exactly one element, and they are different. Without loss of generality $U = \{a\}$ and $V = \{b\}$, and now the result is clear. $\square$

Problem 8 (due Weds 9/18): Let X be a set and fix $x_0 \in X$. Let $\mathcal{T}_{x_0} = \{U \subseteq X \mid x_0 \in U\} \cup \{\emptyset\}$. Prove that $\mathcal{T}_{x_0}$ is a topology.

Solution: We are given that $\emptyset \in \mathcal{T}_{x_0}$, and clearly $X \in \mathcal{T}_{x_0}$ since $x_0 \in X$. Now let $U_\alpha \in \mathcal{T}_{x_0}$ for all $\alpha \in \Lambda$. If all U_α are empty, then their union is empty, hence open. If some U_α is non-empty, it must contain x_0, so the union of all of them contains x_0, and hence is open. Finally, let $U, V \in \mathcal{T}_{x_0}$. If U or V is empty, so is their intersection, hence $U \cap V$ is open. If neither is empty, they both contain x_0, hence so does their intersection, so it is open. $\square$

Problem 9 (due Weds 9/18): Let $X = \mathbb{N}$ with the finite complement topology. Let $\mathcal{B} = \{U \subseteq X \mid 2024 \leq |X \setminus U| < \infty\}$. Prove that $\mathcal{B}$ is a basis for a topology, and the topology it generates equals the finite complement topology.

Solution: Let $x \in X$. Since X is infinite, there exists $S \subseteq X \setminus \{x\}$ with $|S| = 2024$. Set $U = X \setminus S$, so $X \setminus U = S$. Now $U \in \mathcal{B}$, and $x \in U$, thus proving the first basis axiom. Next let $x \in U \cap V$ for $U, V \in \mathcal{B}$. Note that $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$. Since $2024 \leq |X \setminus U|, |X \setminus V| < \infty$, we have $2024 \leq |(X \setminus U) \cup (X \setminus V)| < \infty$, so in fact $U \cap V \in \mathcal{B}$. This proves the second axiom. Now we must show that the topology generated by $\mathcal{B}$ equals the finite complement topology. Every set in $\mathcal{B}$ has finite complement, so one direction is clear. Now let $x \in W$ for an arbitrary $W \subseteq X$ with $|X \setminus W| < \infty$. Since X is infinite, we can choose $S \subseteq X \setminus W$ with $|S| = 2024$. Set $W' := W \setminus S$. Since $x \in W$ we have $x \notin S$, so $x \in W'$. Also, $|X \setminus W'| = |X \setminus (W \setminus S)| \geq |S| = 2024$, so $W' \in \mathcal{B}$. Now $x \in W' \subseteq W$ reveals that every W open in the finite complement topology is also open in the topology generated by $\mathcal{B}$, and we are done. $\square$

No homework due 9/25, just the exam.

Problem 10 (due Weds 10/2): Let $X = \mathbb{N}$ with the finite complement topology and let $S \subseteq X$. Prove that the interior $\mathring{S}$ is empty if and only if $X \setminus S$ is infinite.

Solution: $(\Rightarrow)$: Contrapositively, suppose that $X \setminus S$ is finite. Then S is non-empty and open, and so $\mathring{S} = S$ is non-empty. $(\Leftarrow)$: Contrapositively, suppose $\mathring{S} \neq \emptyset$, so there exists a non-empty open U contained in S. Thus $X \setminus U$ is finite, and hence so is its subset $X \setminus S$. $\square$

Problem 11 (due Weds 10/2): Let X be a topological space in which every subset is either closed or dense. Prove that every subset of X containing a non-empty open set is open.

Solution: Let $\emptyset \neq U \subseteq S \subseteq X$ for U open in X. We claim that S is open. Since $X \setminus S$ is contained in the closed set $X \setminus U \neq X$, the closure of $X \setminus S$ is too, and hence cannot equal X. This means $X \setminus S$ is not dense, so it must be closed, and thus S is open. $\square$

Problem 12 (due Weds 10/2): Let X and Y be [added: non-empty] [oh shoot also not singletons!] sets with the finite complement topology. Prove that if the product topology on $X \times Y$ equals the finite complement topology on $X \times Y$, then X and Y are both finite.

Solution: Suppose X and Y are not both finite, so without loss of generality X is infinite. Fix $y_0 \in Y$ (here we needed to assume the sets are non-empty). Now $Y \setminus \{y_0\}$ is open in Y, so $X \times (Y \setminus \{y_0\})$ is open in the product topology on $X \times Y$. But the complement of $X \times (Y \setminus \{y_0\})$ in $X \times Y$ is $X \times \{y_0\}$, which is infinite [and not all of $X \times Y$ since we rule out X and Y being singletons], so $X \times (Y \setminus \{y_0\})$ is not open in the finite complement topology. $\square$

Problem 13 (due Weds 10/9): Let $f\colon X \to Y$ be a continuous surjection of topological spaces. Suppose that Y is Hausdorff, and that for all $y \in Y$ the subspace $f^{-1}(\{y\})$ of X is open and Hausdorff. Prove that X is Hausdorff.

Solution: Let $x \neq x'$ in X. First suppose $f(x) \neq f(x')$. Since Y is Hausdorff we can choose disjoint open neighborhoods V and V' of $f(x)$ and $f(x')$ in Y. Let $U = f^{-1}(V)$ and $U' = f^{-1}(V')$. Now $x \in U$, $x' \in U'$, U and U' are open in X, and they are disjoint. Alternately, suppose $f(x) = f(x')$, call it y, so $x, x' \in f^{-1}(\{y\})$. Since $f^{-1}(\{y\})$ is Hausdorff, we can choose disjoint open neighborhoods of x and x' in $f^{-1}(\{y\})$, and since $f^{-1}(\{y\})$ is open in X, these neighborhoods are open in X, so we are done. $\square$

Problem 14 (due Weds 10/9): Prove that $C = \prod_{\mathbb{N}}\{0,1\}$ (with the product topology) is Hausdorff.

Solution: Let $(a_1, a_2, \dots) \neq (b_1, b_2, \dots)$ in C, so these are infinite sequences of 0s and 1s. Choose some $i \in \mathbb{N}$ such that $a_i \neq b_i$, say without loss of generality that $a_i = 0$ and $b_i = 1$. Let $U_i = \{0\}$, $V_i = \{1\}$, and $U_j = V_j = \{0,1\}$ for all $j \neq i$. Now $U_1 \times U_2 \times \cdots$ and $V_1 \times V_2 \times \cdots$ are open neighborhoods of $(a_1, a_2, \dots)$ and $(b_1, b_2, \dots)$ respectively, and they are disjoint since the ith entry of any element of their intersection would have to be simultaneously 0 and 1. $\square$

Problem 15 (due Weds 10/9): Let X be a space with the finite complement topology. Prove that if X is Hausdorff then it is finite.

Solution: Suppose X is Hausdorff. Let $x \neq y$ in X. Choose disjoint open neighborhoods U of x and V of y in X. Since U and V are non-empty and open, they have finite complement. Now $U \cap V = \emptyset$ implies $(X \setminus U) \cup (X \setminus V) = X$, so X is a union of two finite sets, hence is finite. $\square$

No homework over October break (nothing due 10/16).

Problem 16 (due Weds 10/23): Let (X, d) be a metric space and let $x \in X$. Let $f_x\colon X \to \mathbb{R}$ be $f_x(y) := d(x, y)$. Prove that f_x is continuous.

Solution: Let (a, b) be a basic open set in $\mathbb{R}$, so $a < b$. Then $f_x^{-1}(a, b) = \{y \in X \mid a < d(x, y) < b\}$. This is equal to $\{y \in X \mid y < b\} \cap \{y \in X \mid a < y\}$. That first set is the basic open set $B_b(x)$, so it suffices to show that $\{y \in X \mid a < y\}$ is open. Indeed, for any y in this set, the open ball centered at y with radius $(y - a)/2$ is fully contained in this set. $\square$

Problem 17 (due Weds 10/23): Let (X, d) be a metric space. Let $Y \subseteq X$. The *induced metric* d_Y on Y is defined to be $d_Y(y, y') := d(y, y')$. (This is clearly a metric, you don't have to prove that.) Prove that the metric topology on Y coming from d_Y equals the subspace topology on Y coming from X.

Solution: First let us prove that every basic open set in the metric topology is open in the subspace topology. A basic open set in the metric topology is of the form $\{y' \in Y \mid d_Y(y, y') < r\}$ for some $y \in Y$ and $r \in \mathbb{R}$. This equals $Y \cap \{x' \in X \mid d(y, x') < r\}$, i.e., Y intersect a (basic) open set in X, so indeed this is open in the subspace topology. Next let us prove that every open set in the subspace topology is open in the metric topology. Let $Y \cap U$ be an arbitrary open set in the subspace topology, so U is open in X. Let $y \in Y \cap U$. Choose $r \in \mathbb{R}$ such that $B_r(y) \subseteq U$ (here $B_r(y)$ is the open ball of radius r centered at y in X). Now $Y \cap B_r(y) = \{y' \in Y \mid d(y, y') < r\} = \{y' \in Y \mid d_Y(y, y') < r\}$, which is a basic open set in the metric topology. Hence $y \in Y \cap B_r(y) \subseteq Y \cap U$ shows that $Y \cap U$ is open in the metric topology. $\square$

Problem 18 (due Weds 10/23): Let X be a set. A function $d\colon X \times X \to \mathbb{R}$ is a *pseudometric* if it satisfies the same axioms as a metric, except we allow $d(x, y) = 0$ even if $x \neq y$. Define $B_r(x) = \{y \in X \mid d(x, y) < r\}$ for $r \in \mathbb{R}$. It turns out the collection of all $B_r(x)$ forms a basis for a topology on X (you don't need to prove this). Prove that if X with this pseudometric topology coming from d is Hausdorff, then d is actually a metric, i.e., $d(x, y) = 0$ implies $x = y$.

Solution: Suppose the pseudometric topology on X is Hausdorff, and suppose $x, y \in X$ with $d(x, y) = 0$. We must show that $x = y$. Since X is Hausdorff, it suffices to show that every (basic) open neighborhood of x intersects every (basic) open neighborhood of y. Let $x \in B_r(z)$ and $y \in B_s(w)$. Since $x \in B_r(z)$ we know $d(z, x) < r$, and since $d(x, y) = 0$ we get $d(z, y) \leq d(z, x) + d(x, y) < r$, so $y \in B_r(z)$. Thus $y \in B_r(z) \cap B_s(w)$, so this is non-empty. $\square$

No homework due 10/30, just the exam.

Problem 19 (due Weds 11/6): Let $X = \mathbb{Z}$ with the "particular point" topology, where a non-empty set is open if and only if it contains 0. Prove that X is path connected.

Solution: Let $x, y \in X$. Let $p\colon [0,1] \to X$ be defined piecewise by $p(0) = x$, $p(t) = 0$ for all $0 < t < 1$, and $p(1) = y$. To prove this is a path from x to y, we need it to be continuous. Let U be an open set in X. If U is empty then its preimage is empty, hence open. Now suppose U is non-empty, so $0 \in U$, and hence $(0,1) \subseteq p^{-1}(U)$. But every subset of $[0,1]$ containing $(0,1)$ is open in $[0,1]$, so p is continuous. □

Problem 20 (due Weds 11/6): Prove that $\mathbb{R}$ with the finite complement topology is path connected. [As a remark, it turns out $\mathbb{Z}$ with the finite complement topology is not path connected, but this is kind of hard to prove.]

Solution: Let $x, y \in \mathbb{R}$, WLOG say $x \leq y$. Let $p\colon [x,y] \to \mathbb{R}$ be $t \mapsto t$. If we can show p is continuous, then it will be a path from x to y. Let C be a closed subset of $\mathbb{R}$ with the finite complement topology, so C is either finite or $\mathbb{R}$. If $C = \mathbb{R}$ then $p^{-1}(C) = [x,y]$ is closed. If C is finite then since p is injective, $p^{-1}(C)$ is finite. Finite subsets of $[x,y]$ (with the standard topology) are closed since $[x,y]$ is Hausdorff, so $p^{-1}(C)$ is closed, and hence p is continuous. □

Problem 21 (due Weds 11/6): Let (X,d) be a metric space. A *geodesic* is a path $p\colon [a,b] \to X$ such that for all $t, t' \in [a,b]$ we have $d(p(t), p(t')) = |t - t'|$. Call X a *geodesic space* if for all $x, y \in X$ there exists a geodesic from x to y. Prove that any geodesic space is locally path connected.

Solution: Let $x \in X$ and let U be an open neighborhood of x. Choose $r > 0$ such that $x \in B_r(x) \subseteq U$. It suffices to prove $B_r(x)$ is path connected, and for this it suffices to show that for all $y \in B_r(x)$ there exists a path in $B_r(x)$ from x to y. Let $p\colon [a,b] \to X$ be a geodesic from x to y, so we just have to prove that $p(t) \in B_r(x)$ for all $t \in [a,b]$. First note that $b - a = d(x,y) < r$. Next observe that $d(x, p(t)) = t - a \leq b - a < r$, so indeed $p(t) \in B_r(x)$. □

Problem 22 (due Weds 11/13): Let X be a Hausdorff space in which every subspace is compact. Prove that X is finite.

Solution: Since every compact subspace of a Hausdorff space is closed, every subspace of X is closed, i.e., X is discrete. The space X itself is a subspace of X, hence is compact. But the only discrete compact spaces are finite. □

Problem 23 (due Weds 11/13): Let X be a non-compact topological space. Prove that there exists a strictly descending chain $X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$ of subspaces all of which are non-compact.

Solution: Since X is non-compact, it is infinite. Let $x \in X$, and we claim $X \setminus \{x\}$ is non-compact. Choose an open cover of X with no finite subcover. If some subcover of $X \setminus \{x\}$ were finite, then adding one more open set from the cover to this subcover, containing x, would yield a finite subcover of X, which cannot exist, so instead there is no finite subcover for $X \setminus \{x\}$. This shows $X \setminus \{x\}$ is non-compact. It is still infinite, so we can repeat this argument to get a strictly descending chain of non-compact subspaces. $\square$

Problem 24 (due Weds 11/13): Let $X = \mathbb{R}$ with the *countable complement topology*, i.e., a subset is open whenever it is empty or has countable complement. (This is obviously a topology.) Is X compact? Prove or disprove. [Hint: It might be useful to remember that every closed subset of a compact space is compact.]

Solution: No. Consider $\mathbb{Z} \subseteq \mathbb{R}$. Every subset of $\mathbb{Z}$ is countable, hence closed in X, hence closed in the subspace topology on $\mathbb{Z}$. This shows that the subspace topology on $\mathbb{Z}$ is discrete. Since $\mathbb{Z}$ is infinite and discrete, it is non-compact. Since $\mathbb{Z}$ non-compact, but closed in X, we conclude that X cannot be compact. $\square$

Problem 25 (due Weds 11/20): Let X be a space with the countable complement topology, as in Problem 24. Prove that if X is second countable then it is countable and discrete.

Solution: Suppose X is second countable, so X has a countable dense subspace Y. Since Y is countable, it is closed in X. Since Y is dense, in fact $Y = X$. Thus, X is countable. Since it has the countable complement topology, it is therefore also discrete. $\square$

Problem 26 (due Weds 11/20): Let $X = \mathbb{Z}$ with the particular point topology (so U is open if $U = \emptyset$ or $0 \in U$). Let $f\colon X \to X$ be the identity map $f(x) = x$, and let $f'\colon X \to X$ be the constant map $f'(x) = 0$. Prove that $f \simeq f'$. [Remark: This proves X is "contractible".]

Solution: Let $F\colon X \times [0,1] \to X$ be defined by sending (x,t) to 0 whenever $t \in (0,1]$ and sending $(x,0)$ to x. Clearly $F(x,0) = f(x)$ and $F(x,1) = f'(x)$ for all $x \in X$, so we just need to prove that F is continuous. Let U be a non-empty subset of X, so $0 \in X$. Then $F^{-1}(U)$ equals $((0,1] \times X) \cup ([0,1] \times U)$, which is open. $\square$

Problem 27 (due Weds 11/20): View S^1 as the unit circle $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Let $X = \mathbb{R}^4 \setminus \{(a,b,c,d) \mid (a,b) = (c,d)\}$. Let $f\colon X \to S^1$ be the map sending (a,b,c,d) to (x,y) whenever the vector in $\mathbb{R}^2$ from (a,b) to (c,d) has the same direction as the (unit) vector from $(0,0)$ to (x,y). Prove that f is not nullhomotopic. [This one might be way too hard, heh.]

Solution: Let $\iota\colon S^1 \to X$ send (x,y) to $(0,0,x,y)$, so ι is continuous and $f \circ \iota = \mathrm{id}_{S^1}$. Since S^1 is not contractible, id_{S^1} is not nullhomotopic, so we conclude that f is not nullhomotopic. $\square$
