Updated November 21, 2024

Homework problems for AMAT 300 (Intro to Proofs), Fall 2024. Over the course of the semester I'll add problems to this list, with each problem's due date specified. Each problem is worth 2 points.

Solutions will be gradually added (and may be hastily written without proofreading).

Problem 1 (due Weds 9/4): Compute the cardinality of the subset $\{2^4, 3^3, 4^2\}$ of $\mathbb{Z}$. Explain your reasoning.

Solution: This set equals $\{16, 27, 16\} = \{16, 27\}$, so the cardinality is 2.

Problem 2 (due Weds 9/4): Prove that $\{3a - 5b \mid a, b \in \mathbb{Z}\} = \mathbb{Z}$.

Solution: ($\subseteq$): Let $3a - 5b$ be an element of the first set, so $a, b \in \mathbb{Z}$. Hence $3a - 5b \in \mathbb{Z}$. ($\supseteq$): Let $z \in \mathbb{Z}$. Set $a = 2z$ and $b = z$. Then $3a - 5b = 6z - 5z = z$, so z is in the first set. $\square$

Problem 3 (due Weds 9/4): Let A and B be sets. Assume that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ (here $\mathcal{P}$ denotes power set). Prove that $A \subseteq B$.

Solution: Let $a \in A$. Then $\{a\} \subseteq A$, so $\{a\} \in \mathcal{P}(A)$. This implies $\{a\} \in \mathcal{P}(B)$, i.e., $\{a\} \subseteq B$, and so $a \in B$ as desired. $\square$

Problem 4 (due Weds 9/11): Let $A = \{(x, x^2 - 4) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ and $B = \{(x, 3x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$. Compute $A \cap B$, and prove rigorously that your answer is right.

Solution: Let $C = \{(4, 12), (-1, -3)\}$. We claim $A \cap B = C$. ($\subseteq$): Let $(x, y) \in A \cap B$, so $y = x^2 - 4$ and $y = 3x$. Hence $x^2 - 4 = 3x$, so $(x - 4)(x + 1) = 0$, and x is either 4 or -1. In these respective cases, y is 12 or -3, so we conclude that $(x, y) \in C$. ($\supseteq$): We must show that $(4, 12), (-1, -3) \in A \cap B$. Indeed, $4^2 - 4 = 12$ and $(-1)^2 - 4 = -3$. $\square$

Problem 5 (due Weds 9/11): Let A and B set sets. Prove (rigorously) that $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$.

Solution: ($\subseteq$): Let $x \in (A \cup B) \setminus (A \cap B)$, so $x \in A \cup B$ but $x \notin A \cap B$. If $x \in A$ then $x \notin B$, so $x \in A \setminus B$. Alternately, if $x \in B$ then $x \notin A$, so $x \in B \setminus A$. In either case, $x \in (A \setminus B) \cup (B \setminus A)$. ($\supseteq$): Let $x \in (A \setminus B) \cup (B \setminus A)$, so either $x \in A \setminus B$ or $x \in B \setminus A$. In the first case, $x \in A$ and $x \notin B$, so $x \in A \cup B$ and $x \notin A \cap B$. In the second case, $x \in B$ and $x \notin A$, so $x \in A \cup B$ and $x \notin A \cap B$. In either case, we conclude that $x \in (A \cup B) \setminus (A \cap B)$. $\square$

Problem 6 (due Weds 9/11): Let X and Y be sets. Prove that $\mathcal{P}(X \cap Y) = \mathcal{P}(X) \cap \mathcal{P}(Y)$.

Solution: ($\subseteq$): Let $A \in \mathcal{P}(X \cap Y)$, so $A \subseteq X \cap Y$. Then $A \subseteq X$ and $A \subseteq Y$, i.e., $A \in \mathcal{P}(X)$ and $A \in \mathcal{P}(Y)$, so $A \in \mathcal{P}(X) \cap \mathcal{P}(Y)$. ($\supseteq$): Let $A \in \mathcal{P}(X) \cap \mathcal{P}(Y)$, so $A \in \mathcal{P}(X)$ and $A \in \mathcal{P}(Y)$. Thus, $A \subseteq X$ and $A \subseteq Y$, i.e., $A \subseteq X \cap Y$, so $A \in \mathcal{P}(X \cap Y)$. $\square$

Problem 7 (due Weds 9/18): Let A and B be sets. Prove that

$$\overline{(A \cup B) \setminus (A \cap B)} = (\overline{A} \cap \overline{B}) \cup (A \cap B).$$

[Hint: It's easier to use DeMorgan's Laws than to do the usual element argument for proving set equality.]

Solution: Applying DeMorgan repeatedly, we have

$$\overline{(A \cup B) \setminus (A \cap B)} = \overline{(A \cup B) \cap \overline{A \cap B}} = \overline{(A \cup B)} \cup (A \cap B) = (\overline{A} \cap \overline{B}) \cup (A \cap B)$$

as desired. $\square$

Problem 8 (due Weds 9/18): Let $I = (0, 1)$. For each $\alpha \in I$ let $A_\alpha = (\alpha, \alpha + 1)$. Prove (rigorously) that $\bigcup_{\alpha \in I} A_\alpha = (0, 2)$. [Just to be clear, all these "$(blah, blah)$" things are intervals in $\mathbb{R}$, not ordered pairs in $\mathbb{R}^2$.]

Solution: ($\subseteq$): Let $x \in \bigcup_{\alpha \in I} A_\alpha$, so $x \in A_\alpha$ for some $\alpha \in I$. This means $\alpha < x < \alpha + 1$ and $0 < \alpha < 1$. Hence $0 < x < 2$, i.e., $x \in (0, 2)$. ($\supseteq$): Let $x \in (0, 2)$, so $0 < x < 2$. Set $\alpha_0 := x/2$. Note that $0 < \alpha_0 < 1$, so $\alpha_0 \in I$. Also, $\alpha_0 < x$, and $x - 1 = 2\alpha_0 - 1 < \alpha_0$, so we conclude that $x \in (\alpha_0, \alpha_0 + 1)$. This shows $x \in A_{\alpha_0}$, and hence x is in the union of all the A_α. $\square$

Problem 9 (due Weds 9/18): Let $I = (0, 1)$. For each $\alpha \in I$ let $A_\alpha = (\alpha, \alpha + 1)$. Prove (rigorously) that $\bigcap_{\alpha \in I} A_\alpha = \{1\}$.

Solution: ($\subseteq$): Let $x \in \bigcap_{\alpha \in I} A_\alpha$, so $x \in (\alpha, \alpha + 1)$ for all $\alpha \in (0, 1)$. Since $x > \alpha$ for all $\alpha < 1$, we see that $x \geq 1$. Since $x < \alpha + 1$ for all $\alpha > 0$, we see that $x \geq 1$. Hence $x = 1$. ($\supseteq$): We must show that $\alpha < 1 < \alpha + 1$ for all $0 < \alpha < 1$, but this is immediate. $\square$

No homework due 9/25, just the exam.

Problem 10 (due Weds 10/2): Prove that the statement $P \Rightarrow Q$ is logically equivalent to the statement $P \Rightarrow (P \Rightarrow Q)$.

Solution: Make their truth tables, observe they're the same. $\square$

Problem 11 (due Weds 10/2): Prove that x being an element of $\mathbb{Q}$ is enough to ensure that there is a natural number n with $nx \in \mathbb{Z}$. [First convert this into a purely "logical" statement, then prove it.]

Solution: $\forall x \in \mathbb{Q} \exists n \in \mathbb{N}$ such that $nx \in \mathbb{Z}$. Proof: Let $x \in \mathbb{Q}$, say $x = p/q$ for $p, q \in \mathbb{Z}$. Note that $q \neq 0$. Up to possibly replacing p/q with $(-p)/(-q)$ we can assume $q > 0$, i.e., $q \in \mathbb{N}$. Now $n = q$ satisfies $nx = q(p/q) = p \in \mathbb{Z}$. $\square$

Problem 12 (due Weds 10/2): Let X be a set. Prove that $|X \times X| \leq 31$ if and only if $|X| \leq 5$.

Solution: ($\Rightarrow$): Suppose $|X \times X| \leq 31$, so $|X| \cdot |X| \leq 31$, so $|X| \leq \sqrt{31}$. Since $|X|$ is a whole number, this implies $|X| \leq \sqrt{25} = 5$. ($\Leftarrow$): Suppose $|X| \leq 5$. Then $|X| \cdot |X| \leq 25 \leq 31$, so $|X \times X| \leq 31$. $\square$

Problem 13 (due Weds 10/9): Prove (rigorously) that if $x \in \mathbb{Z}$ is odd then $x^2 + 7x - 4$ is even.

Solution: Let $x \in \mathbb{Z}$ be odd, say $x = 2n + 1$ for some $n \in \mathbb{Z}$. Then $x^2 + 7x - 4 = (2n+1)^2 + 7(2n+1) - 4 = 4n^2 + 18n + 4 = 2(2n^2 + 9n + 2)$, and $2n^2 + 9n + 2 \in \mathbb{Z}$, so this is even. $\square$

Problem 14 (due Weds 10/9): Let A and B be non-empty sets. Prove that if $A \times B = B \times A$ then $A = B$.

Solution: Suppose $A \times B = B \times A$. Let $a \in A$. Since $B \neq \emptyset$ we can choose $b \in B$. Now $(a, b) \in A \times B$, which since $A \times B = B \times A$ implies $(a, b) \in B \times A$, hence $a \in B$. This shows $A \subseteq B$, and an analogous argument shows $B \subseteq A$, so $A = B$. $\square$

Problem 15 (due Weds 10/9): For sets A and B, prove that if $A \neq B$ then $A \cup B \neq A \cap B$. [Hint: Using the contrapositive is probably best.]

Solution: Suppose $A \cup B = A \cap B$. Let $a \in A$. Then $a \in A \cup B$, so $a \in A \cap B$, hence $a \in B$. This shows $A \subseteq B$, and an analogous argument shows $B \subseteq A$, so $A = B$. $\square$

No homework over October break (nothing due Weds 10/16)

Problem 16 (due Weds 10/23): Prove (rigorously) that if $x^3 - 5x^2 + 8x - 4 \geq 0$ then $x \geq 1$.

Solution: Suppose $x < 1$, so $(x-1) < 0$. By inspection, $x = 1$ is a root of $x^3 - 5x^2 + 8x - 4$, so we can factor out $(x-1)$ and get $x^3 - 5x^2 + 8x - 4 = (x-1)(x^2 - 4x + 4) = (x-1)(x-2)^2$. Since $(x-2)^2 \geq 0$ and $(x-1) < 0$ we have $x^3 - 5x^2 + 8x - 4 < 0$. $\square$

Problem 17 (due Weds 10/23): Prove that there do not exist $a, b \in \mathbb{Z}$ satisfying $2024 \cdot a - 2013 \cdot b = 1$.

Solution: Suppose a and b do exist. Since 2024 and 2013 are both divisible by 11, so is $2024 \cdot a - 2013 \cdot b$. But then 1 is divisible by 11, a contradiction. $\square$

Problem 18 (due Weds 10/23): Let $x \in \mathbb{N}$ with $x > 3$. Prove that x, $x+2$, and $x+4$ cannot all be prime. [Hint: You can use the fact that every $n \in \mathbb{Z}$ is of the form $3m$, $3m+1$, or $3m+2$, for some $m \in \mathbb{Z}$.]

Solution: First suppose $x = 3m$ for some $m \in \mathbb{Z}$. Since $x > 3$, this shows x is divisible by 3 but does not equal 3, so x is not prime. Next suppose $x = 3m+1$ for some $m \in \mathbb{Z}$. Then $x + 2 = 3m + 3 = 3(m+1)$, so $x+2$ is not prime. Finally, suppose $x = 3m+2$ for some $m \in \mathbb{Z}$. Then $x + 4 = 3m + 6 = 3(x+2)$ is not prime. $\square$

No homework due 10/30, just the exam.

Problem 19 (due Weds 11/6): Use induction to prove that $7 | (29^n - 8)$ for all $n \in \mathbb{N}$. [Advice: If some products of big numbers arise, don't bother calculating them, just leaving them written as products will make things easier later.]

Solution: The base case $n = 1$ holds since $29^1 - 8 = 21$, which is divisible by 7. Now suppose $n \geq 2$, and assume for the induction hypothesis that $7 | (29^{n-1} - 8)$. Say $29^{n-1} - 8 = 7m$ for some $m \in \mathbb{Z}$. Then $29^n - 8 \cdot 29 = 7m \cdot 29$, so $29^n - 8 = 7m \cdot 29 + 8 \cdot 28$. This equals $7(29m + 32)$, which is divisible by 7. $\square$

Problem 20 (due Weds 11/6): Use strong induction (with two base cases) to prove that $6 | (n^3 - n)$ for all $n \in \mathbb{N}$.

Solution: For two base cases, note that $1^3 - 1 = 0$ and $2^3 - 2 = 6$, both of which are divisible by 6. Now suppose $n \geq 3$, and assume for an induction hypothesis that $6 | (m^3 - m)$ for all $1 \leq m < n$. In particular $6 | ((n-2)^3 - (n-2))$, say $(n-2)^3 - (n-2) = 6k$ for some $k \in \mathbb{Z}$. Computing this out we get $n^3 - 6n^2 + 11n - 6 = 6k$, so $n^3 - n = 6k + 6n^2 - 12n + 6$, which equals $6(k + n^2 - 2n + 1)$, hence is divisible by 6. $\square$

Problem 21 (due Weds 11/6): Prove that for any $n \in \mathbb{N}$ with $n \geq 18$, one can pay exactly n cents in postage using only 4-cent and 7-cent stamps.

Solution: As four base cases, note that $18 = 7+7+4$, $19 = 7+4+4+4$, $20 = 4+4+4+4+4$, and $21 = 7+7+7$. Now suppose $n \geq 22$, and assume for induction that for all $18 \leq m < n$, we can achieve m cents. In particular this works for $m = n-4$, say $n - 4 = 4k + 7\ell$. Then $n = 4(k+1) + 7\ell$, so we can also achieve n cents. $\square$

Problem 22 (due Weds 11/13): Let R be the relation on $\mathbb{Z}$ defined by: xRy whenever $x + y = 3^k$ for some $k \in \mathbb{N}$. Prove that R is symmetric, but neither reflexive nor transitive.

Solution: Suppose xRy. Then $x+y = 3^k$ for some $k \in \mathbb{N}$, so $y+x = 3^k$, so yRx. This shows R is symmetric. To see that R is not reflexive, note that $0 + 0 = 0$ does not equal 3^k for some $k \in \mathbb{N}$, so $0R0$ is false. To see that R is not transitive, note that $1 + 2 = 3^1$ so $1R2$, and $2 + 7 = 3^2$ so $2R7$, but $1+7 = 8$ is not of the form 3^k for any $k \in \mathbb{N}$, so $1R7$ is false. $\square$

Problem 23 (due Weds 11/13): Let R be the relation on $\mathbb{N} \times \mathbb{N}$ defined by: $(a, b)R(c, d)$ whenever $a^2 + d^2 = b^2 + c^2$. Prove that R is an equivalence relation.

Solution: Note that this is equivalent to $a^2 - b^2 = c^2 - d^2$ [which makes this one easier than I realized]. Now the reflexive and symmetric properties are immediate since $a^2 - b^2 = a^2 - b^2$ and if $a^2 - b^2 = c^2 - d^2$ then $c^2 - d^2 = a^2 - b^2$. For the transitive property, suppose $a^2 - b^2 = c^2 - d^2$ and $c^2 - d^2 = e^2 - f^2$. Then $a^2 - b^2 = e^2 - f^2$. $\square$

Problem 24 (due Weds 11/13): Let A be a set and R a relation on A that is symmetric and transitive. For each $a \in A$ let $[a] = \{b \in A \mid aRb\}$. Prove that R is an equivalence relation if and only if $[a] \neq \emptyset$ for all $a \in A$.

Solution: ($\Rightarrow$): Suppose R is an equivalence relation. Let $a \in A$. Since R is reflexive, aRa, so $a \in [a]$, so $[a] \neq \emptyset$. ($\Leftarrow$): Suppose $[a] \neq \emptyset$ for all $a \in A$. Since R is symmetric and transitive, we just need to prove it is reflexive. Let $a \in A$. Since $[a] \neq \emptyset$, we can choose $b \in [a]$, so aRb. By the symmetric property, bRa, and then by the transitive property aRa, so we conclude that R is reflexive. $\square$

Problem 25 (due Weds 11/20): Let $f\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(x, y) = 3x-2y$. Prove that f is surjective but not injective.

Solution: Let $z \in \mathbb{Z}$. Set $x = z$ and $y = z$. Then $f(x, y) = 3x - 2y = 3z - 2z = z$. We conclude f is surjective. To see f is not injective, note that $(0, 0) \neq (2, 3)$ but $f(0, 0) = 0$ and $f(2, 3) = 0$. $\square$

Problem 26 (due Weds 11/20): Let X be a set and let R be an equivalence relation on X. Let $f\colon X \to \mathcal{P}(X)$ be the function defined by $f(x) = [x]$ where $[x]$ is the equivalence class of x with respect to R. Prove that if f is injective then R is just the relation "=".

Solution: Suppose f is injective. Suppose xRy, and we must show that $x = y$. Since xRy we have $[x] = [y]$, so $f(x) = f(y)$. Since f is injective, $x = y$. $\square$

Problem 27 (due Weds 11/20): Let $f\colon A \to B$ be a surjective function. For each $b \in B$ let $H(b) = \{a \in A \mid f(a) = b\}$, so intuitively $H(b)$ is the "horizontal line" at b. Prove that $\{H(b) \mid b \in B\}$ is a partition of A.

Solution: Since f is surjective, for all $b \in B$ there exists $a \in A$ with $f(a) = b$, and hence $a \in H(b)$. This shows each $H(b)$ is non-empty. For any $a \in A$, if $b = f(a)$ then $a \in H(b)$, so the union of the $H(b)$ is all of A. Lastly, suppose $H(b) \cap H(b') \neq \emptyset$, say $a \in H(b) \cap H(b')$. Then $f(a) = b$ and $f(a) = b'$, so $b = b'$, and hence $H(b) = H(b')$. This shows that any two distinct $H(b)$ are disjoint, so it is a partition. $\square$